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A new look at dimensional analysis from a group theoretical viewpoint

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Abstract. The dimensionalisation hypothesis given in a previous paper is modified and established in a way which is as independent of arbitrary choices as possible. The process of group contraction is also analysed as well as the relation between dimensions in the original and contracted group. The theory is illustrated with the example of Cayley-Klein geometries and an ultrarelativistic limit of the Poincaré group.

1. Introduction

A few years ago we proposed a group theoretical support for dimensional analysis in the context of kinematic groups (Cariñena *et al* 1981). The main idea was to trace back to the kinematic group structure the 'dimensional analysis' (DA) for the world described by that group. Although the description of spacetime as an homogeneous space of a kinematic group is known to be only approximate (for classical gravitation, a general semi-Riemannian 4-manifold against the flat Minkowski space), the results of such a study will hopefully retain their significance in the general case as we have a Poincaré group of isometries in each tangent space. It could also be useful for other theories (such as gauge theories) involving other groups.

The meaning of the results obtained in our previous paper by means of the 'dimensionalisation hypothesis' needs no further emphasising. There we started not with an abstract Lie algebra with some generators A_i, but instead, each A_i had a very concrete physical meaning (i.e. a generator of space translations along a line, of pure inertial transformations, etc). However a drawback to this treatment was pointed out in the previously quoted paper: 'not every element of the algebra has a dimension (e.g. $H+P_1$ has not)' (Cariñena *et al* 1981, p 7). This is very unsatisfactory because $H+P_1$ is as worthy a Lie algebra element as H or P_1, and in fact, in the Galilei group, the subgroup generated by $H+P_1$ is conjugated to the subgroup generated by H because we have (in the Galilei group)

$$\exp(vK_1)\exp H\exp(-vK_1)=\exp(H+vP_1).$$

On the basis of this formula we could argue that, as H and P_1 have different dimensions, H and P_1 need a relative *dimensional* coefficient to be added. In fact, as v has dimension LT^{-1}, each each element in $H+vP_1$ has, according to the standard values of DA, a dimension of T^{-1}, a result which is in agreement with the physical meaning of $H+vP_1$ as generating time translations relative to a reference system which is moving with a speed $-v$ along the 1-axis from our original reference system.

The drawback of this procedure appears more evident when we realise that this kind of reasoning has to be made *after* the dimensionalisation hypothesis has been used for giving dimension to some Lie algebra elements, i.e. the elements of the chosen 'physical' basis. If the process is repeated starting from some other basis, nothing assures us that the results will remain unchanged. In fact, simple examples show that the results do indeed change. Take as an example the $1+1$ Poincaré group, which in the physical basis H, P, K is

$$[K, H]=P \qquad [K, P]=H \qquad [H, P]=0$$

and refer it to a 'light-cone basis' $\{P_+, P_-, K\}$ where

$$P_+=\tfrac{1}{2}(H+P) \qquad P_-=H-P$$

with Lie brackets

$$[K, P_+]=P_+ \qquad [K, P_-]=-P_- \qquad [P_+, P_-]=0.$$

If we apply the dimensionalisation hypothesis naively we obtain $[K]=1$ and $[P_+]$, $[P_-]$, two *independent* dimensions. Whether this has some meaning or not, this result clearly conflicts with the results obtained in an ordinary way, i.e. P and H have the same dimension (which could naively be expected to be that of P_+ and P_-), because if we had started with the light-cone basis, we had to find an argument showing that $H=P_++\frac{1}{2}P_-$ and $P=P_+-\frac{1}{2}P_-$ have the same dimension.

Another simple example is provided by the two-dimensional Euclidean group, referred to the bases $A=J$, $B=J+P_1$, $C=J+P_2$. Here the naive application of the DA meets more difficulties, because the commutator of two basis elements is not a simple basis element, e.g. $[A, B]=C-A$. If one insists in assigning dimensions to A, B, C in such a way that *all* non-zero structure constants are dimensionless, this would lead to A, B, C dimensionless. This is of course both reasonable and adequate, as A, B, C generate rotations around different points, but the drawback is now that there is no trace of the presence in the group of generators as $B-A$ or $C-A$ which are *not* dimensionless in the standard treatments.

After mentioning some of the difficulties appearing in the theory proposed in the earlier paper depending on the initial choice of the basis, the benefits are clear for examining again more carefully its 'dimensionalisation hypothesis', looking for a method of establishing it in a way as independent of arbitrary choices as possible and trying to find an explanation for the success of the hypothesis with the particular choices given in the paper. This is one of the motivations for the present paper, but we do not restrict ourselves to the study of kinematic groups as in the earlier paper but we also consider another classical problem, that of nine plane geometries, for which a summary can be found in the excellent (and amusing) book by Yaglom (1979). Concerning problems more directly related to physics we shall analyse an 'ultrarelativistic contraction' of the Poincaré group. As a by-product of the theory of dimensional analysis presented here, we obtain a new physical interpretation of the mathematical process of group contraction as well as the relation between dimensions in the original and contracted group, which is far more interesting when performing non-natural changes which go to natural changes in the limit defining the contraction.

The scheme of this paper is as follows. In § 2 we present a short summary of Cayley-Klein geometries and their groups which are going to be used later as examples of the developed theory, pointing out its geometrical properties and natural relationship by means of different kinds of geometrical contractions, duality, etc. In the same light we present an unusual contraction of the $3+1$ Poincaré group related to the 'ultrarelativistic limit' of relativistic mechanics. In § 3 we present a new development of DA in the context of group theory, remarking on its new characteristics when compared with the previous one. Finally § 4 is devoted to the discussion of some examples, ranging from the one-dimensional conformal geometry to the 'ultrarelativistic' group introduced in § 2.

2. Some Cayley–Klein geometries

The so-called two-dimensional (or plane) Cayley-Klein geometries historically appeared as geometries subordinate to the projective (space)geometry in the works of Cayley and Klein. Furthermore, under the name quadratic, they were studied by Poincaré in a paper on the fundamental hypotheses of the geometry (see, for example, Torreti (1978, p 180) and references therein). In that work Poincaré furnished all the quadratic geometries with a common axiomatic foundation based on the assumption that the 'plane' (i.e. the set of 'points') is a two-dimensional differentiable manifold, whose set of motions is a three-dimensional Lie group acting on the 'plane'.

In modern terms, these geometries appear as G spaces for some three-dimensional Lie groups acting on some of their two-dimensional homogeneous spaces in the normal way. The geometries so obtained are locally homogeneous by construction. In the following we will give a brief presentation of these geometries and we do not pay attention to the topological (global) properties (i.e. we do not distinguish between spherical and elliptic geometry) and so we can work with Lie algebras in order to simplify some technical points.

Consider the Lie groups SO(3), SO(2, 1), E(2), $\mathcal{M}(2)$ and $\mathcal{G}(2)$, the last two standing for Minkowskian and Galilean in $1+1$ dimensions, respectively. All these groups have a three-dimensional Lie algebra generated, say, by A, B, C with Lie brackets given by

Table 1.

	SO(3)	SO(2, 1)	E(2)	$\mathcal{M}(2)$	$\mathcal{G}(2)$
$[C, A]$	B	B	B	B	B
$[C, B]$	$-A$	$-A$	$-A$	0	0
$[A, B]$	C	$-C$	0	$-C$	0

Next, we consider for each group G a one-dimensional subgroup N and the corresponding two-dimensional homogeneous space G/N which plays the role of 'plane'. The elements of N leave invariant the point $\mathrm{O} \equiv \{\mathrm{N}\}$ in the plane, and may be considered as playing the role of 'rotations' around O. Now we have to single out (straight) lines in our geometry. This can be done by several different methods (e.g. by defining them as the autoparallel curves of the canonical connection (Kobayashi and Nomizu 1963, p 300) in G/N which exists, provided some extra conditions hold here. This procedure, however, has some disadvantages in relation to Cayley-Klein geometries). We proceed as follows: in all the geometries we consider, to every line l is associated a one-dimensional subgroup of G whose elements play the role of 'translations' along l. Conversely, if such a subgroup $\mathcal{T}$ is designated as 'translations along l', then l can be considered as the trace of any of its points by means of $\mathcal{T}$. Any subgroup conjugated to $\mathcal{T}$ will correspond to translations along another line. So a 'complete' set of lines is obtained by choosing a one-dimensional subgroup $\mathcal{T}$ which is taken to be the one 'generating' a chosen line l such that $\mathrm{O} \in l$. Every two-dimensional Cayley–Klein geometry fits in this scheme for adequate selections of G, N and $\mathcal{T}$. From the Lie algebra viewpoint, the selection of N and $\mathcal{T}$ amounts to selecting the generators denoted by J and H. A third basis element of the Lie algebra will be called P; the transformations generated by P will be referred to as 'special translations' and they can be either ordinary translations or not (this is to be discussed separately in each case), in the same way as rotations can also be translations in some cases (e.g. in SO(3)).

Table 2 shows how the nine plane Cayley–Klein geometries follow from this scheme. In each case we give the selections of J, H and P in the Lie algebra, as well as the Lie brackets $[J, H]$, $[J, P]$ and $[H, P]$. For some geometries the selections are given in two (equivalent) forms for reasons that we shall explain later (see, for example Fernández Sanjuán 1984).

A remarkable relationship between these geometries is provided by duality, which interchanges 'points' with 'lines'. At the level of the Lie algebra, duality is thus described

Table 2.

Spherical geometry $G=SO_3(\mathbb{R})$ $J=C,\ H=A,\ P=B$ P, $-H$, J P, $-H$, J $J=A,\ H=-C,\ P=B$	Euclidean geometry $G=E(2)$ $J=C,\ H=A,\ P=B$ P $-H$ 0	Hyperbolic geometry $G=SO(2,1)$ $J=C,\ H=A,\ P=B$ P $-H$ $-J$
Co-Euclidean geometry $G=E(2)$ $J=A,\ H=-C,\ P=B$ P 0 J	Galilean geometry $G=\mathcal{G}(2)$ $J=C,\ H=A,\ P=B$ P, 0, 0 P, 0, 0 $J=A,\ H=-C,\ P=B$	Co-Minkowskian geometry $G=\mathcal{M}(2)$ $J=C,\ H=A,\ P=B$ P, 0, $-J$ P, 0, $-J$ $J=-B,\ H=A,\ P=-C$
Co-hyperbolic geometry $G=SO(2,1)$ $J=A,\ H=-C,\ P=B$ P H J	Minkowskian geometry $G=M(2)$ $J=A,\ H=-C,\ P=B$ P, H, 0 P, H, 0 $J=A,\ H=B,\ P=-C$	Doubly hyperbolic G $G=SO(2,1)$ $J=-B,\ H=A,\ P=-C$ P, H, $-J$ P, H, $-J$ $J=A,\ H=B,\ P=-C$

by the transformation

$$J \to H \qquad H \to -J \qquad P \to P.$$

This can be seen as follows: a geometrical meaning is given by choosing a one-dimensional subgroup, generated by J, to be interpreted as rotations around O, i.e. leaving fixed a point O, and another one-dimensional subgroup, generated by H, which will be interpreted as translations along a line, i.e. leaving invariant a line l. Now, if we take a different choice with the new J playing the role of the old H, we shall consider as the 'point origin' the geometric object preserved by H, namely the line l; similar comments concern the new with respect to the old J. Finally, the choice of signs is just a matter of convenience.

Duality corresponds to 'reflection on the main diagonal' in table 2. Some geometries are self-dual (e.g. for spherical geometry this corresponds to the already known polarity which interchanges 'pole' with 'equator'), but others are not. The prefix 'co' applied to some geometries corresponds to this fact ('anti' is also used by other authors (Schober 1981)). Of course the groups of two dual geometries are isomorphic.

What is more interesting from the physical viewpoint is the relationship between these geometries provided by the idea of an IW (Inonu and Wigner 1953) contraction. As is well known, an IW contraction of a group G is fully determined by the 'non-contracted' subgroup S. As we have points and lines as outstanding objects of the geometry, we have two IW contractions with a clear geometrical meaning by taking either the isotopy subgroup of a point or that of a line, respectively, for S.

In the first case we obtain a *point-like*, or *local*, contraction, given by the replacement

$$P \to \varepsilon P \qquad H \to \varepsilon H \qquad J \to J$$

and then putting $\varepsilon \to 0$. In this contraction each geometry goes to its 'middle column neighbour', which near a point 'looks' like the original geometry. This kind of contraction is well known and carries the spherical and hyperbolic geometries into the Euclidean one.

In the second case, we obtain a different contraction, *line-like* or *axial*, given by

$$P \to \varepsilon P \qquad J \to \varepsilon J \qquad H \to H$$

and then $\varepsilon \to 0$. Now each geometry goes to its 'middle row neighbour' which 'looks' like the original geometry near a line. This kind of contraction is the one carrying the Minkowskian into the Galilean geometry, as its study was the starting point for IW. Nonetheless, its geometrical meaning is very clear but not usually stated.

The simultaneous consideration of all these geometries related by the two kinds of contractions is very convenient for some purposes. It is perhaps worth noticing that only the upper and lower rows appear as, respectively, two-dimensional Riemannian and pseudo-Riemannian spaces of constant curvature, but the middle row does not appear in this way because the candidate for 'metric' is singular; this fact accounts for many of the peculiarities of the non-relativistic theories as opposed to the corresponding relativistic ones, both from the mathematical and the physical viewpoints.

The hyperbolic geometry is very interesting in some of its aspects. We recall that the motions in this geometry can be classified into three kinds (according to the three conjugation classes of one-parameter subgroups in SO(2, 1), depending on whether the motion has either only one point, or one or two points but at infinity, as fixed points. The corresponding motions are then called elliptic (or rotation), parabolic (or horocyclic displacement) and hyperbolic (or translation), respectively. In the first two cases the fixed point is referred to as the centre of the motion: in the third the line determined by the two fixed points is called the axis of the translation. A very convenient model for visualising the hyperbolic geometry is the so-called Poincaré disc conformal model (Pedoe 1970), in which the points at infinity are represented by the set complex numbers of modulus one.

A known and very interesting property of hyperbolic geometry is that the whole group of motions still acts in a faithful way on the set of points at infinity (this is not so in the Euclidean case). This is connected with the existence of a one-dimensional homogeneous space for the group of hyperbolic motions, obtained by quotient by a two-dimensional subgroup, which is unique up to conjugations (see, for example, Stowe 1983). In order to see more clearly this connection, we perform a slight generalisation of the idea of duality. Let us consider hyperbolic geometry, and tentatively call 'points' in a new geometry the pencil of hyperbolic parallels, i.e. the set of all lines passing through a fixed point at infinity ω (lines with end ω according to Hilbert (1971)). The role of rotations will be taken for the set of transformations having fixed a given 'point' of the new geometry, i.e. by the set of hyperbolic motions which apply a given pencil of hyperbolic parallels onto itself. It is easy to see that the set contains horocyclic displacements with centre ω and translations along any line in the pencil, and coincides with the subgroup generated by horocyclic displacements with centre ω, and by translations along a given line through ω. With the generators of hyperbolic motions as given in table 1, the point ω can be taken as the point 'at $+\infty$' on the line generated from O by the subgroup $\exp(bH)$, and hence horocyclic displacements with their centre at this point are generated by $K = C - B = J - P$; for translations along a line through the given point at infinity we choose $A = H$ itself. As expected, A and K close a two-dimensional algebra, $[A, K] = K$, and the set of points at infinity can be represented as the corresponding homogeneous space of the group of hyperbolic motions.

From the physical viewpoint, this geometry appears as being the geometry of light-ray 'directions' for light propagation in a plane, according to relativity theory, because free motion in a plane of massive particles correspond to ordinary points in the hyperbolic geometry of uniform motions (Juárez and Santander 1982) so that points 'at infinity' correspond to free motions of zero mass.

In the Poincaré disc model if $\omega = 1$ is chosen, then it is easy to obtain expressions for the action of motions on the points at infinity $\{e^{i\beta}\}$. Here one can see another interesting property of this set, that this manifold is homeomorphic to a circle, and hence compact.

Sometimes the so-called conformal group in one dimension is given as the group generated by the following transformations on the one-point compactification of the real line:

$$t \to t + b \qquad b \in \mathbb{R} \tag{2.1a}$$

$$t \to e^{-\lambda} t \qquad \lambda \in \mathbb{R} \tag{2.1b}$$

$$t \to \frac{t}{1 - \alpha t} \qquad \alpha \in \mathbb{R}. \tag{2.1c}$$

The generators of these three one-parameter subgroups, say T, D and K, are given by

$$T = -\frac{d}{dt} \qquad D = +t\frac{d}{dt} \qquad K = -t^2\frac{d}{dt} \tag{2.2}$$

and close a Lie algebra

$$\begin{aligned} [D, K] &= K \\ [D, T] &= -T \\ [T, K] &= 2D \end{aligned} \tag{2.3}$$

which is easily recognised as a SO(2, 1) Lie algebra; in fact the correspondence

$$D \leftrightarrow A \qquad T \leftrightarrow C + B \qquad K \leftrightarrow C - B$$

displays the standard form of the SO(2, 1) Lie algebra (table 2). The relations (2.1) define the action of the hyperbolic group SO(2, 1) acting on a one-dimensional homogeneous space, and hence this action must be equivariant to the one on the points at infinity in the hyperbolic plane; in fact, it is easy to see that the relation between these transformation groups, is given (if one considers the Poincaré disc model with points at infinity $e^{i\beta}$) by

$$e^{i\beta} \to t = \tan \tfrac{1}{2}\beta.$$

Here 'translations' (2.1*a*) in the variable t correspond to horocyclic displacements around $e^{i\beta} = -1$, generated by $J + P = C + B$; 'dilatations (2.1*b*) around $t = 0$' correspond to translations along the line with ends -1 and 1, generated by H, and finally 'special conformal transformations' (2.1*c*) correspond to the horocyclic displacements around $e^{i\beta} = 1$, generated by $J - P = C - B$.

The correspondence $t = \tan \frac{1}{2}\beta$ has an immediate geometrical meaning, as it corresponds to a stereographic projection of the circle of radius 1 in the (complex) Euclidean plane (which plays the role of points at infinity in the Poincaré disc model of a hyperbolic plane) from the point $z = -1$ on the imaginary axis. Using known facts about inversions in the Euclidean plane it is very easy to understand, at least in qualitative terms, the correspondences hereafter referred to.

It is interesting to know what the transformations of the variable t are corresponding to rotations around O and translations along the line through O generated by P. Very easy calculations give the results

$$t \to \frac{t + \tan \frac{1}{2}\theta}{1 - t \tan \frac{1}{2}\theta} = \frac{t \cos \frac{1}{2}\theta + \sin \frac{1}{2}\theta}{\cos \frac{1}{2}\theta - t \sin \frac{1}{2}\theta} \tag{2.4a}$$

and

$$t \to \frac{t + \tanh \frac{1}{2}\chi}{1 - t \tanh \frac{1}{2}\chi} = \frac{t \cosh \frac{1}{2}\chi + \sinh \frac{1}{2}\chi}{\cosh \frac{1}{2}\chi + t \sinh \frac{1}{2}\chi} \tag{2.4b}$$

respectively. The first subgroup, with generator $R = -(1 + t^2)\, d/dt$, is not conjugated to the subgroup of dilatations around $t = 0$. It is perhaps noteworthy that the difference between 'translations' and special conformal transformations in the t variable is only nominal, because they are conjugated (e.g. by means of the transformation $t \to (t+1)/(1-t)$, which corresponds to a rotation of angle π around the origin in the hyperbolic plane).

Another geometry of Cayley-Klein type which will be considered in this paper is an 'ultrarelativistic' contraction of the 3 + 1 Poincaré group. As far as we know, this group has not been considered earlier, but its study could perhaps be useful for high-energy problems and also in relation to the group theoretical treatment of geometrical optics (Bacry 1984). A related group (but not exactly the same) has been proposed for the description of the ultrarelativistic limit by Quirós and Ramírez-Mittelbrun (1981).

The group $\mathcal{U}$ we are proposing is simply the IW contraction of the Poincaré group $\mathcal{P}$ to the neighbourhood of a *light-like line* (in a similar way, the Galilei group is a *time-like line* contraction of $\mathcal{P}$). We will choose the light-like line $x=(x_1, x_2)=0$, $x_3=t$ (the actual choice is irrelevant) and then the stability subgroup of that line is the subgroup generated by J_3, K_3, P_+ and E_i ($i=1,2$). The notation is that of the Poincaré algebra in the 'null plane' basis as given by Leutwyler and Stern (1978):

$$P_+=\tfrac{1}{2}(P_0+P_3) \qquad E_i=\tfrac{1}{2}(\varepsilon_{ij}J_j-K_i).$$

An adequate complement for this set for obtaining a basis of $L\mathcal{P}$ is P_-, P_i, F_i ($i=1,2$), given by

$$P_-=P_0-P_3 \qquad F_i=-\varepsilon_{ij}J_j-K_i.$$

The IW contraction we are considering will be given by the replacement

$$\begin{array}{llll} J_3\to J_3 & K_3\to K_3 & P_+\to P_+ & E_i\to E_i \\ P_-\to \varepsilon P_- & P_i\to \varepsilon P_i & F_i\to \varepsilon F_i & \end{array}$$

and then taking the limit of ε going to zero.

The non-zero brackets of the Lie algebra $L\mathcal{P}$ and its contraction $L\mathcal{U}$ are given in the following, where the subindices go from 1 to 2. The symbol (0) refers to the Lie brackets which go to 0 in the contraction:

$$\begin{array}{lll} [K_3, E_r]=E_r & & \\ [J_3, E_r]=\varepsilon_{rs}E_s & & [J_3, P_i]=\varepsilon_{ij}P_j \\ [P_i, E_r]=\delta_{ir}P_+ \quad (0) & & \\ [K_3, P_+]=P_+ & & \\ [F_r, E_r]=-\delta_{rs}K_3-\varepsilon_{rs}J_3 \quad (0) & & [F_r, K_3]=F_r \\ [F_r, P_+]=-P_r & & [F_r, J_3]=-\varepsilon_{rs}F_s \\ [P_-, E_r]=P_r & & [F_r, P_i]=-\delta_{ri}P_- \quad (0) \\ & & [P_-, K_3]=P_-. \end{array}$$

The fact that the Poincaré Lie algebra contains subalgebras isomorphic to the Lie algebra of the extended 2+1 Galilei group is known (Elizalde and Gomis 1976). In the null-plane basis this can easily be seen: both sets $\{P_-, P_i, -E_i, J, P_+\}$ and $\{P_+, P_i, -F_i, J, P_-\}$ generate a 2+1 extended Galilei algebra, with these generators being the time and space translations, pure inertial transformations, rotations and 'phase' subgroup. We notice that in the group $\mathcal{U}$ these sets even close algebras if we could remove P_+ or P_-, respectively, so that in the contracted group we obtain (as one would naturally expect) subalgebras isomorphic to a 2+1 Galilei Lie algebra. We notice that in the 'ultrarelativistic' group proposed by Quirós and Ramírez-Mittelbrun this fact does not occur but instead of this we would obtain a 2+1 Carroll Lie algebra (Lévy-Lebond 1965).

3. Dimensional analysis

When a group G is considered as a transformation group acting transitively on some space X, and so is endowed with a geometric meaning, the natural candidates to primitive geometrical magnitudes are just the canonical parameters of its one-parameter subgroups. For instance, for plane Euclidean geometry, these candidates are just the lengths and angles. In the general case we cannot expect, however, such a direct association of canonical parameters with invariant quantities for each pair of points, pair of lines, etc, but they will always have some meaning. Other parameters, such as the area, are related to more complex constructs over X and will not be considered here.

If we take two elements g, g' in the same one-dimensional subgroup, the value of the quotient of its canonical parameters is well determined; this value could then be taken as a 'measure' of the element g' when g is considered as a 'unit'. The choice of g as a 'unit' amounts to selecting a given element A in the Lie algebra $\mathscr{G}$ of G related to g by $g=\exp(A)$, i.e. g corresponds to the value of 1 of the canonical parameter. A change of the unit g in the given one-dimensional subgroup is implemented by a change $A \to \lambda A$, with $\lambda \neq 0$, of the corresponding generator and we will usually refer to A as the 'unit' for its one-parameter subgroup, Of course, for each one-dimensional subgroup unrelated units can be selected. But it seems to be better to:

(*a*) investigate to what extent units for some one-parameter subgroups can be 'propagated' for the others, and

(*b*) realise if there are some 'natural' changes of the unit systems. This is the aim of the present research.

Well known examples suggest to us the use of conjugation as a method for 'propagating' units, because conjugate subgroups have the 'same' geometrical interpretation. In order to put this idea into formal terms, let us consider the action of G on the one-parameter subgroups of G by inner automorphisms. As a consequence of the relation $g(\exp X)g^{-1}=\exp(\mathrm{Ad}_g X)$, two generators A and B are in the same orbit if there exists $g \in \mathrm{G}$ such that $\mathrm{Ad}_g A = B$. We shall denote by $A \sim B$ the corresponding equivalence relation.

We also consider the action of G on the one-dimensional subgroups. Here, as B and λB, with $\lambda \neq 0$, generate the same one-dimensional subgroup, the relevant equivalence relation, to be denoted by $\approx$, is given by $A \approx B$ iff there exist $g \in \mathrm{G}$, and $\lambda \in \mathbb{R}$, $\lambda \neq 0$, such that $\mathrm{Ad}_g A = \lambda B$. When speaking of the equivalence class of A, without more specifications, we shall understand it to mean the $\approx$ class.

The idea of the propagation of units as taken from the conjugation can be translated as follows: 'if A has been chosen as a unit for its one-dimensional subgroup, and $A \sim B$, then select B as a unit for its subgroup'. This idea works provided that the one-dimensional subgroup $\{\exp(tA)\}$ has no non-trivial self-conjugations. In fact, let us suppose $X \sim Y$, i.e. there exists $g \in \mathrm{G}$ such that $g\exp(tX)g^{-1}=\exp(tY)$. If there exists another $g' \in \mathrm{G}$ such that $g' \exp(tX)g'^{-1} = \exp(t\lambda Y)$, with $\lambda \neq 1$, $\lambda \in \mathbb{R}^*$, then one should take Y and λY simultaneously as units in the same subgroup, because both are obtained by 'propagation' from X by means of a conjugation. It is trivial to show that in this case $g^{-1}g'$ is a non-trivial self-conjugation of the one-dimensional subgroup $\exp(tX)$.

The case where the one-dimensional subgroup $\exp(tA)$ has a one-parameter subgroup of self-conjugations can be easily identified in the Lie algebra of G. Let us suppose that X generates such a subgroup; as $\exp(\mu \mathrm{Ad} X)A = \lambda(\mu)A$, and $\lambda(\mu_1+\mu_2) = \lambda(\mu_1)+\lambda(\mu_2)$ we obtain $\lambda(\mu)=\mathrm{e}^{\alpha\mu}$ with $\alpha \neq 0$. By means of a derivation in $\mu=0$ we obtain $(\mathrm{Ad}X)A = \alpha A = [X, A]$. Then there exists an element $D = \alpha^{-1}X$ in the Lie algebra such that $[D, A] = A$. Conversely, if there exists such a D the elements of the one-parameter subgroup generated by D are self-conjugations of $\exp(tA)$.

In order to see the structure here more clearly, let $\mathscr{C}$ denote the set of the elements of a $\approx$ class. The set $\mathbb{R}^*$ of non-zero real numbers acts naturally on $\mathscr{C}$ as given by $\lambda: B \to \lambda B$. Each orbit is simply a 'line' (without the point O) in $\mathscr{C}$ and the orbit space is to be denoted by $\mathcal{O}$. Then $\mathscr{C}$ has a natural bundle structure of base $\mathcal{O}$ and fibre $\mathbb{R}^*$. We could now consider an action of $\mathbb{R}^* \otimes \mathrm{G}$ in $\mathscr{C}$ given as follows: $(\lambda, g): B \to \lambda \mathrm{Ad}_g B$. This action is transitive because of the factor $\mathbb{R}^*$. The subgroup G (as identified with $1 \otimes \mathrm{G}$) acts on $\mathscr{C}$, and there are two possibilities:

(i) This action is transitive. Then the subgroup $\{\exp(tA)\}$ for $A \in C$ has non-trivial self-conjugations; $\mathscr{C}$ is homeomorphic to some homogeneous space for G.

(ii) This action is not transitive. Then we do not have any non-trivial self-conjugation of $\{\exp(tA)\}$ and we can consider an action of G on $\mathcal{O}$ by projection. This last action will always be transitive. Here $\mathcal{O}$ is an homogeneous space for G.

In the first case, $\mathscr{C}$ itself is homeomorphic to G/S, where S is the subgroup of G such that $s \in \mathrm{S}$ if $\mathrm{Ad}_s A = A$ for any A in the class $\mathscr{C}$; in the second $\mathcal{O}$ is homeomorphic to G/S, where now $s \in \mathrm{S}$ if $\mathrm{Ad}_s A = \lambda A$, $\lambda \in \mathbb{R}^*$.

For the Lie algebras of Cayley–Klein geometries and others of low dimension this situation can easily be depicted and gives most of the homogeneous spaces for these groups.

The preceding discussion can be summarised as follows. There are two essentially different situations. For classes $\mathscr{C}$ where there are no non-trivial self-conjugations, a good intrinsic transport of units can be defined by means of the $\sim$ relation. Two generators have the same 'measure' iff they are on the same $\sim$ class. This class will be called of the first kind. For the other classes, to be called the second-kind classes, the transport of A as a unit to other conjugated subgroups depends on the choice of a section of the bundle $\mathscr{C} \to \mathcal{O}$. Notice however that the quotient of the measures of two elements in the same subgroup, which are the significant values, does not depend on this section. This section will be referred to as a transport of units.

A transport of units having been selected whenever necessary, we denote by $A \equiv B$, the equivalence relation given by 'B is obtained from A by the transport of units'. By this method we can obtain a *unit system* for G (i.e. for all of its one-parameter subgroups). Its ingredients are:

(i) A set of elements A_α, one for each $\approx$ class, α being a class index.

(ii) For each of the second-kind classes, a concrete selection of the conjugation carrying $\exp(tA)$ onto every one-dimensional conjugated subgroup (a section in the corresponding bundle $\mathscr{C} \to \mathcal{O}$).

From now on, we shall consider the 'measure' of the elements X in the Lie algebra in the following way. In the subgroup $\exp(tX)$ there is a unique unit, which can be obtained by means of the transport from the unit $A_{\alpha(X)}$ initially selected in the $\approx$ class $\alpha(X)$ of X. Then there is a well determined non-zero real number κ such that $X \equiv \kappa A_{\alpha(X)}$. The number κ will be taken as the measure of X relative to the given unit system.

Now we turn our attention to point (b) in order to analyse if there are some 'special' changes of the unit systems. According to the viewpoint developed above, a change in a unit system consists of both a change in each primitive unit A_α and a change of the chosen transport whenever necessary. In this paper we shall only consider the change of the primitive units, but we leave unchanged the transport because the correct interpretation of the change of this transport is not yet fully clear to us.

Let us choose a basis of the Lie algebra $\mathscr{G}$, $\{X_i\}$, $i=1,\dots,m$. Relative to this basis (whose choice is generally a matter of convenience for the problem at hand) we could classify the classes into two subsets:

(i) those of the m elements X_i as well as those of the non-null Lie brackets $[X_i, X_j]$ (we shall call them *basic* classes (with respect to $\{X_i\}$); and

(ii) all the other classes.

A natural definition of adaptation or suitability for the basis is the following: 'when two elements X_i and X_j are in the same class, then $X_i \equiv X_j$'. From any arbitrary basis a new adapted basis can always be obtained by means of simple scale changes in some of the X_i. Such a basis will be called *adapted* to the unit system.

For every non-zero Lie bracket $[X_i, X_j]$ we have a non-zero real number κ_{ij} such that

$$[X_i, X_j] \equiv \kappa_{ij} A_{\alpha([X_i,X_j])}$$

where of course $A_{\alpha([X_i,X_j])}$ is the primitive unit in the class of $[X_i, X_j]$. We now consider the change of units given by $A_\alpha \to A'_\alpha = \lambda(\alpha)A_\alpha$, with a non-zero real number $\lambda(\alpha)$ for each class (but keeping fixed the same transport), and the corresponding change of the (adapted) basis $X_i \to X'_i = \lambda(\alpha(X_i))X_i$. If we define a new set of numbers κ'_{ij} by means of the corresponding relation it is an easy matter to check that the κ_{ij} and κ'_{ij} are related by the following equation

$$\kappa'_{ij} = \kappa_{ij} \frac{\lambda(\alpha(X_i))\lambda(\alpha(X_j))}{\lambda(\alpha[X_i, X_j])}. \tag{3.1}$$

Among all the primitive unit system changes, we pick out the so-called *natural* ones. By definition, a natural change of units is a change of units for which the linear transformation of the algebra given by

$$X_i \to X'_i = \lambda(\alpha(X_i))X_i \tag{3.2}$$

is an automorphism.

The restriction of being an automorphism of the Lie algebra can be important. In fact, only if X is Ad-nilpotent may there be an automorphism ϕ_λ of $\mathscr{G}$ such that $\phi_\lambda(X) = \lambda X$ for any $\lambda \in \mathbb{R}$. This property is based on the following lemma.

Lemma. A linear automorphism ϕ of the vector space of $\mathscr{G}$ is an automorphism of $\mathscr{G}$ if and only if $\mathrm{Ad}\phi(X) = \phi \circ \mathrm{Ad}X \circ \phi^{-1}$, $\forall X \in \mathscr{G}$.

Proof. Let Y be an arbitrary element of $\mathscr{G}$ and $Z = \phi(Y)$. Then, if we take into account that $\mathrm{Ad}\phi(X)Z = [\phi(X), \phi(Y)]$ and $\phi \circ \mathrm{Ad}X \circ \phi^{-1}(Z) = \phi([X, Y])$ we obtain the result of the lemma.

Proposition. If for any $\lambda \in \mathbb{R}$ there is an automorphism ϕ_λ of the Lie algebra $\mathscr{G}$ such that $\phi_\lambda(X) = \lambda X$, then X is Ad-nilpotent.

Proof. If there is such an automorphism then $\lambda\,\mathrm{Ad}X = \phi_\lambda \circ \mathrm{Ad}X \circ \phi_\lambda^{-1}$ according to the result of the lemma and therefore $\mathrm{Ad}X$ and $\lambda\,\mathrm{Ad}X$ have to have the same invariant factors which is only possible if $\mathrm{Ad}X$ is nilpotent.

The meaning of the natural changes is clearly seen in the familiar case of Euclidean geometry. The standard basis J, P_1, P_2 is adapted, and a unit system is given by, say, J and P_1. From the general change $J \to \alpha J$, $P_1 \to \lambda P_1$ the automorphsim condition implies that $\alpha = 1$. This corresponds to a well known property of that geometry: the (numerical) relations in a given situation do not change explicitly when the unit of length is changed, but do change in an explicit way if the angular unit is changed. In general, the numerical relations concerned in a given problem will remain exactly identical for two unit systems related by a natural change. The case of the plane hyperbolic geometry is very interesting, because there are no natural changes. If one insists in performing a change of, say, the unit of length, the numerical relations will use explicitly a constant whose value depends on the unit of length chosen. It is perhaps worth remembering that this peculiar fact hindered the acceptance of absolute geometry in its early days (Bonola 1955, p 89).

For a natural change of units, the scale factors $\lambda(\alpha)$ of the basic classes are related by means of the equations $\lambda(\alpha(X_i))\lambda(\alpha(X_j)) = \lambda(\alpha[X_i, X_j])$ whenever $[X_i, X_j] \neq 0$. These relations can also be stated by saying that natural changes leave the values κ_{ij} unchanged.

Given a basis X_i in the Lie algebra $\mathscr{G}$, the set of all the natural changes has a group structure, isomorphic to $(\mathbb{R}^+)^n$ for some natural number n. This is easily seen in the following way. Let us consider a real linear space $V_p \approx \mathbb{R}^p$ whose dimension p coincides with the number of basic classes. For each possible change of units given by factors $\lambda(\alpha)$, let us consider the real numbers $Ln\lambda(\alpha)$ as components of a vector in the canonical basis of $\mathbb{R}^p$. Then the set of the equations for natural changes becomes a set of linear equations for the numbers $Ln\lambda(\alpha)$, which determines a subspace N_n of dimension $n \leqslant p$, of $V_p \approx \mathbb{R}^p$. If a basis $\{\theta_a\}$, $a = 1, 2, \ldots, n$, is chosen in N_n, then every element in N_n can be given by a linear combination of θ_a, and hence

$$Ln(\lambda(\alpha)) = \sum_{a=1}^{n} (Ln\gamma^a)\theta_{a,\alpha}$$

where $\theta_{a,\alpha}$ denote the components of θ_a in the canonical basis of $\mathbb{R}^p$ and the coefficients in the linear combination have been written as $Ln\gamma^a$. Then

$$\lambda(\alpha) = \exp\left(\sum_{a=1}^{n} (Ln\gamma^a)\theta_{a,\alpha}\right) = \prod_{a=1}^{n} (\gamma^a)^{\theta_{a,\alpha}}$$

and therefore the numbers γ^a can be considered as the parameters of a natural change, relative to the basis θ_a, and the real numbers $\theta_{a,\alpha}$ play the role of dimensional coefficients of the class α in the basis $\{\theta_a\}$.

Notice the resemblance of the relations we have obtained to the conventional ones in DA (Hulin 1981) that in the case we have discussed have a group theoretical support. In particular, it is now clear that the term 'dimension of X_i' is meaningless; the correct way is to speak of the 'dimensions (or dimensional coefficients) of X_i with respect to the basis θ_a'. Nevertheless, if a basis θ_a has been chosen we could simplify matters by speaking of the 'dimension' of X_i (i.e. one set of dimensional coefficients).

Under natural changes, the numbers κ_{ij} (which are tantamount to structure constants and *are* exactly structure constants in some particular cases) *do not change* their values. We could even say that the κ_{ij} are *dimensionless* under natural changes. The actual numerical value of κ_{ij} can appear in a formula or relation, but this value being invariant under natural changes, the same formula or relation will result after a natural change has been performed. In the particular case where the κ_{ij} are the structure constants, the natural changes are actually those determined by the 'dimensionalisation hypothesis' of our earlier paper. This fact explains the success of this hypothesis in the case of kinematic groups in the 'physical' basis (Cariñena *et al* 1981).

Let us suppose that we do consider a non-natural unit change. The numbers κ_{ij} do not yet behave as dimensionless and change with a factor as indicated previously. If we want to maintain in this case the language used for natural changes, we can say that each κ_{ij} has acquired a 'dimension' which governs its scale change factor under the changes considered. In this case any relation will make the values of the κ_{ij} appear explicitly, but of course the rule of change of the κ_{ij} ensures that no result is affected by these 'spurious' constants, because their explicit values are only a device for dealing with non-natural changes. The mere existence of the κ_{ij} is the relevant point, and has been duly taken into account by the structure of natural changes.

A particular instance where the preceding situation appears is in the case of group contractions. In fact, the analytical description of an IW contraction of groups corresponding in geometrical terms to the 'restriction' to the neighbourhood of a point, a line, etc, requires a non-natural unit change as a way of introducing a 'constant' in terms of whose value a limit $\to\infty$ or $\to 0$ could be considered. The actual meaning of such a 'variation of constants' has been clearly discussed by Lévy-Leblond (1977). The case $c\to\infty$ in the transition to the Galilei group from the Poincaré group is prototypical, and Lévy-Leblond points out the fact that a second 'limit' $c\to\infty$ could be obtained, the Carroll group (Lévy-Leblond 1965). In this case the difficulty is mainly academic, but for other situations not so well understood, the problem of deciding how to insert the constants which have to go to zero or infinity is not trivial and as the preceding example shows, two completely different limits can be obtained. To the examples we have given in our previous paper we could add those relevant to the search for an 'ultrarelativistic limit' of the Poincaré group. The DA we have developed provides a well defined framework where all the questions concerning such problems could be unambiguously answered.

We have also noticed that we have determined natural changes only after a basis had been chosen. This is related to the fact that nothing has been said about the effect of natural changes on the non-basic classes. This is not unsatisfactory because any element in the group can be expressed as a product of elements in the subgroups generated by X_i, and once units have been chosen for the basic classes we can, in principle, avoid completely the explicit mention of units in these non-basic classes, which amounts to a (conventional) choice for the units of non-basic classes in terms of those chosen for the basic classes. We hope to discuss this point as the problem of the role of the 'transport' in a future paper.

4. Examples

In this section we discuss with some examples how to apply the theory developed in § 3. We will make use of the groups of the two-dimensional Cayley–Klein geometries as well as the group introduced in § 2.

For the case of the Euclidean plane group E(2) there are just two $\approx$ classes, the first one being built up by the linear combinations of A and B, both of the first kind. So, using the notation of table 1, a unit system is given by $\{A, C\}$. The natural changes are given by $A\to\lambda A$, $C\to C$ and for them $B\to\lambda B$. So the group of natural changes is $\mathbb{R}^+$ and we find the conventional result: only one 'dimensional' magnitude. When understood in terms of the *Euclidean* geometry this magnitude is the parameter of translations and special transformations, i.e. length, while when understood in terms of the *non-Euclidean* geometry, natural changes correspond to changes in the measurement of angles and the parameter of special transformations, but lengths along lines remain unchanged. This is clear if one thinks in terms of duality.

The SO(3) group is almost trivial from this viewpoint. There is only one $\approx$ class, of the first kind, and there are no natural changes. Hence, all the generators are 'dimensionless'. This, of course, corresponds to the known facts in spherical geometry.

The group SO(2, 1) is a bit more interesting: in the basis given in table 1, there are three classes, with representatives C, A and $K = C - B$, respectively. The classes of C and A are of the first kind, but that of K is of the second kind (as we pointed out before $[A, K] = K$). Let us first consider the basis A, B, C. It is easy to see that the basic classes are those of C and A. There are no natural changes relative to this basis. The meaning of this result is clearly stated with reference to the hyperbolic geometry because the constant *curvature* of the space allows and imposes a definite relation between lengths and angles (by means of the parallelism angle). If a non-natural change in the unit of length is performed (i.e. $A \to \lambda A$), then the structure constant κ_{AB} will acquire a (spurious) dimension $[A]^{-2}$. From a Euclidean viewpoint, we could say that in the hyperbolic plan there is a 'universal constant', whose 'Euclidean dimension' is the square of a length, appearing explicitly in the formulae. The actual meaning of this spurious dimension is that even if in the local contraction hyperbolic geometry goes into Euclidean geometry, and consequently the structure constant κ_{AB} goes to zero, before taking the limit (i.e. for small enough neighbourhoods of a given point), whereas the metrical (i.e. angles and distances) relationships are up to first order identical to those in the limit theory, there remain some second-order discrepancies or anomalies due to curvature as measured, for example, by the Diquet or Bertrand–Puiseux formula (Spivak 1979), and these discrepancies have, in the *limit* theory, a natural dimension of square length. A more interesting example is that of the group $\mathcal{U}$.

The study of the geometry of the points at infinity in the hyperbolic plane from this viewpoint is also enlightening. If we simply translate the results obtained for the former case, C and A are, naturally enough, dimensionless. By the convention we have used, K can also be taken as dimensionless and so for all the generators.

As an application of the theory developed in § 3, the basic classes for the case of the conformal group in one dimension are those of $\{D\}$ and $\{T, K\}$ and the last is of the second kind. As a unit system we can choose D and T, and suppose that a transport is chosen in such a way that the basis is adapted, so that $T = K$. (This is possible if the transport is chosen as being $\exp[\alpha(T + K)]$, i.e. not in the homogeneous subgroup, but not if we are restricted to the homogeneous subgroup as we will comment on later.) Then, we obtain immediately that there are *no* natural changes.

The same result can be obtained by taking another basis. In fact, any conformal transformation can also be decomposed as a product of elements in the one-parameter subgroups (it is called the Iwasawa decomposition of SL(2, $\mathbb{R}$) (see, for example, Wawrzynczyk 1984))

$$t \to \frac{t + \tan\frac{1}{2}\theta}{1 - \tan\frac{1}{2}\theta} \qquad t \to e^{-\lambda t} \qquad t \to \frac{1}{1 - \alpha t} \tag{4.1}$$

generated, respectively, by $R \leftrightarrow C$, $D \leftrightarrow A$ and $K \leftrightarrow C - B$, with commutation relations

$$[D, R] = K - R \qquad [D, K] = K \qquad [R, K] = D. \tag{4.2}$$

Now the basic classes are three classes, those of $\{D, K - R\}$, $\{R\}$ and $\{K\}$, the last being of the second kind. We choose D, R, K as a unit system, and for the application of the theory, the transport for the class of K is not strictly necessary because the basis is adapted no matter what the transport. The theory in § 3 leads to the non-existence of natural changes, in agreement with our previous results.

However, by proceeding in a naive way, and starting with the expressions for the group action, we will obtain *different* results. If we start with the relations (2.1) it is natural to give dimensions, say $[t]$, to the unique variable t; then one is led to assign dimension $[t]$ to the parameter b, $[t]^{-1}$ to α and λ is dimensionless; so, the generators have dimensions, say, $[T]=L^{-1}$, $[K]=L$, $[D]=1$. When looked upon from the viewpoint of the associated hyperbolic geometry $[D]=1$ means that translations are dimensionless, as expected, but the relations $[T]=L^{-1}$, $[K]=L$ assign *different* (although perhaps spurious) dimensions to two *conjugated* generators.

Had we started with the relations (4.1), the same reasoning would lead to $[t]=1$ and then all the parameters are dimensionless. So this naive treatment does not give an unambiguous prescription for the dimensionalisation.

The same results are obtained by working through with our old dimensionalisation hypothesis (Cariñena *et al* 1981). In fact, in the commutation relations we shall assign an unrelated symbol to each generator, demanding that the structure constants be dimensionless, and we shall obtain

$$[D]=1 \qquad [T][K]=1$$

in the basis (2.1). Instead of this we would find

$$[D]=1 \qquad [K]=[R] \qquad [R][K]=1 \qquad \text{so that } [R]=[K]=1$$

when using the basis (4.1).

The origin of the (spurious) dimensions appearing in the first case is linked with the fact that one cannot compare H and K unless a transport has been selected. If this transport is not given, then it is not completely unreasonable to consider $[T]$ and $[K]$ as *a priori* unrelated, and this leads to the dimension $[T]=[K]^{-1}$. From another viewpoint, instead of the formula $e^{i\beta}\to t=\tan\frac{1}{2}\beta$, we can also take $e^{i\beta}\to t=\lambda\tan\frac{1}{2}\beta$ with λ any non-zero real constant and then the (apparent) dimension of t is only a device for a (bad) parametrisation by $\mathbb{R}$ of a set homeomorphic to S^1 (which does not admit 'dilatations') after removing a point (i.e. at the price of introducing a point in which the parametrisation is not regular).

There are other two-dimensional Cayley-Klein geometries, those associated with the groups $\mathscr{G}(2)$ and $\mathscr{M}(2)$. They are left to the reader. Notice only that the case of Minkowskian geometry has some second-kind classes, and hence has similar features to those previously discussed. These features will also appear in our next example.

As a final example we deal with the ultrarelativistic contraction of the Poincaré group proposed in § 2. We first consider the Poincaré group. In the null-plane basis, the basic classes are those of $\{J_3\}$, $\{K_3\}$, $\{E_1, E_2, F_1, F_2\}\{P_+, P_-\}$ and $\{P_1, P_2\}$. The second-kind classes are those of $\{E_1, E_2, F_1, F_2\}$ and $\{P_1, P_2\}$.

A unit system is given by J_3, K_3, E_1, P_+, P_1 and the choice of transports. The natural changes are those given by

$$J_3\to J_3 \qquad K_3\to K_3 \qquad E_1\to E_1 \qquad P_+\to\lambda P_+ \qquad P_1\to\lambda P_1.$$

Then in the Poincaré group we recover the usual statement.

We only have one 'dimension' symbolised by $\mathbb{L}$, common to P_+, P_-, P_i, and all the other basic elements are dimensionless. (Notice that the only use of the old dimensionalisation hypothesis leads to different results.)

We are now going to study the group contraction $\mathcal{U}$. Here the structure of the classes is richer, and the basic classes are those of $\{J_3\}$, $\{K_3\}$, $\{E_1, E_2\}$, $\{F_1, F_2\}$, $\{P_+\}$, $\{P_-\}$. Then a unit system is given by J_3, K_3, E_1, F_1, P_+, P_-, P_1 and the natural changes are those given by

$$J_3 \to J_3 \qquad K_3 \to K_3 \qquad E_i \to rE_i \qquad F_i \to sF_i \qquad P_+ \to tP_+$$
$$P_- \to uP_- \qquad P_i \to vP_i$$

where the positive real numbers r, s, t, u, v fulfil the relations

$$st/v = 1 \qquad ru/v = 1.$$

Then we have five 'dimensions', those of the classes $\{E_i\}$, $\{F_i\}$, $\{P_+\}$, $\{P_-\}$, $\{P_i\}$, respectively symbolised by H^{-1}, G^{-1}, C^{-1}, D^{-1}, L^{-1}, linked by the two relations $G = LC^{-1}$, $H = LD^{-1}$. So we can take L, C, D as 'primitive dimensions' corresponding to the 'basis' automorphisms whose factors $\lambda(\alpha) \equiv (r, s, t, u, v)$ are respectively $(\lambda, \lambda, 1, 1, \lambda)$, $(1, 1/\sigma, \sigma, 1, 1)$ and $(1/\rho, 1, 1, \rho, 1)$. The most general natural change is the general product $(\lambda/\rho, \lambda/\rho, \sigma, \rho, \lambda)$.

Then the 'decoupling' of space and time which is characteristic of the transition from $\mathcal{P}$ to its Galilean limit occurs here between the 'transverse space', whose translations are generated by P_i, and the two 'directions' P_+ and P_-. This could be expected because in the contraction from $\mathcal{P}$ and $\mathcal{U}$ two Galilei $(2+1)$ subalgebras appear, the role of time translations and pure inertial transformations being taken by P_-, $-E_i$ and P_+, $-F_i$, respectively. The dimensional structure obtained can be considered as a natural consequence of the preceding remark and the two 'directions' P_+, P_- could be imagined as some kind of 'time direction'.

Now let us use the theory in order to see the correct way of implementing the rescaling of variables that gives rise to the contracted group through a numerical limit. Notice that the Lie brackets going to zero are $[P_i, E_r]$, $[F_i, E_r]$ and $[P_i, F_r]$. The corresponding coefficients κ_{ij} are dimensionless for natural changes. The main point is now that if one performs not an arbitrary non-natural change in $\mathcal{P}$ but precisely a non-natural change in $\mathcal{U}$ which in the limit is a natural change in the contracted group $\mathcal{U}$, these coefficients acquire a (spurious) dimensionality. We will denote this spurious dimension in $\mathcal{P}$ by the same symbol as in the group $\mathcal{U}$. Thus it is easy to see that the (spurious) dimension of the relevant (i.e. going to zero) coefficients is always $L^2C^{-1}D^{-1}$ which would be the dimension of the corresponding κ_{ij} in the contracted group, *where it is not equal to zero.* The same discussion can be carried out for the more elementary case of the contraction from the hyperbolic to the Euclidean geometry. In the strict viewpoint of Euclidean geometry there is no 'absolute' way of measuring lengths but if we suppose that the *exactly true* geometry is not Euclidean but hyperbolic, and we continue with our Euclidean habits of changing the unit of length, we will describe the *curvature* of space by means of some magnitude K of dimension L^{-2}. This is, of course, a spurious dimension from the viewpoint of hyperbolic geometry and the description of this situation by means of the Euclidean geometry is more and more precise, the more the magnitudes appearing with (Euclidean) dimension L^2 are negligible in front of K.

This discussion can be translated to our case by comparing the dimensions of the generators in $\mathcal{P}$ and $\mathcal{U}$ and we see that one can identify $\mathbb{L}$ with L, and then one needs *two* constants, κ and θ, with (spurious in $\mathcal{P}$) dimensions LD^{-1} and LC^{-1} in order to pass from $\mathcal{P}$ to $\mathcal{U}$. The *values* of κ and θ depend on the unit system we have chosen in $\mathscr{P}$ but under natural changes (in $\mathscr{P}$) these values do not change, and hence the criterion of validity of the application of the group $\mathscr{U}$ to a given situation is as follows. All the magnitudes, that under non-natural changes in $\mathscr{P}$ of the kind (3.2) have dimension $L^2D^{-1}C^{-1}$, ought to be negligible with regard to the product $\kappa\theta$.

Then κ and θ play the role of 'hidden universal constants' in the sense of Lévy-Leblond (1977). We can even use a particular unit system in which their *values* will be 1 (i.e. measure, for example space translations and *light-like* translations, all in metres). In any case, the contraction $\mathscr{P} \to \mathscr{U}$ will be described as the 'limit' whose exact meaning was explained before.

We give the explicit form of the commutation relations for the Poincaré group generators after a non-natural change of the type (3.2) has been performed. The factors (r, s, t, u, v) of such a change are $(1/\kappa, 1/\theta, \theta, \kappa, 1)$ and then the only commutators where κ and θ appear explicitly are

$$[P_i, E_r] = \frac{1}{\kappa\theta}\delta_{ir}P_+ \qquad [F_r, P_i] = \frac{-1}{\kappa\theta}\delta_{ri}P_- \qquad [P_i, E_r] = \frac{-1}{\kappa\theta}(\delta_{rs}K_3 + \varepsilon_{rs}J_3)$$

which in the 'limit' $\kappa\theta \to 0$ give rise to the contracted group $\mathscr{U}$. The fact that the contraction can be reached alternatively with $\kappa \to \infty$, $\theta = ct$ and $\kappa = ct$, $\theta \to \infty$ corresponds to some duality between the sets $\{E_r, P_+\}$ and $\{F_r, P_-\}$ in the Poincaré group, and is probably somewhat linked with the duality for the electromagnetic field in the absence of charges and currents. The Inonu–Wigner contraction with factors $(1, \varepsilon, 1, \varepsilon, \varepsilon)$ is a product of an automorphism (i.e. a natural change) characterised by $(1, 1, \varepsilon, \varepsilon, \varepsilon)$ times a non-natural change with $\theta = 1/\varepsilon$, and $\kappa = 1$, $(1, \varepsilon, 1/\varepsilon, 1, 1)$. The group $\mathscr{U}$ could also be obtained as the limit $\varepsilon \to 0$ of

$$F_i \to \varepsilon F_i \qquad P_+ \to (1/\varepsilon)P_+$$

just in the same way as the Galilei group could also be obtained in the limit $\varepsilon \to 0$ of $\boldsymbol{K} \to \varepsilon\boldsymbol{K}$, $H \to (1/\varepsilon)H$ instead of the geometrically clear contraction $\boldsymbol{K} \to \varepsilon\boldsymbol{K}$, $\boldsymbol{P} \to \varepsilon\boldsymbol{P}$.

Using the results obtained we easily see the rescaling of parameters in the group $\mathscr{P}$ needed to safely reach $\mathscr{U}$ through a 'limit' $\kappa\theta \to 0$. These are given in our table below.

Generators	Parameters in $\mathscr{P}$	Dimension in $\mathscr{P}$	Dimension in $\mathscr{U}$	Scaling	Parameters in $\mathscr{U}$
J_3	ϕ	1	1	—	ϕ
K_3	β	1	1	—	β
$\boldsymbol{E}$	$\boldsymbol{\xi}$	1	LD^{-1}	$\boldsymbol{\xi} = \boldsymbol{v}/\kappa$	$\boldsymbol{v}$
$\boldsymbol{F}$	$\boldsymbol{\zeta}$	1	LC^{-1}	$\boldsymbol{\zeta} = \boldsymbol{w}/\theta$	$\boldsymbol{w}$
P_+	μ	$\mathbb{L}$	C	$\mu = \theta m$	m
P_-	υ	$\mathbb{L}$	D	$\upsilon = \kappa n$	n
$\boldsymbol{P}$	$\boldsymbol{a}$	$\mathbb{L}$	L	—	$\boldsymbol{a}$

Then from the action of $\mathscr{P}$ on spacetime, by means of rescaling and limiting, the group $\mathscr{U}$ appears as a transformation group of a four-dimensional 'light-like' spacetime, homeomorphic to $\mathbb{R}^4$ and naturally parametrised by four coordinates $(r, s, \boldsymbol{x})$, as follows:

	$\exp(\phi J_3)$	$\exp(\beta K_3)$	$\exp(\boldsymbol{v}\cdot\boldsymbol{E})$	$\exp(\boldsymbol{w}\cdot\boldsymbol{F})$	$\exp(mP_+)$	$\exp(nP_-)$	$\exp(\boldsymbol{a}\cdot\boldsymbol{P})$
r	r	$e^{-\beta}r$	r	r	r	$r+n$	r
s	s	$e^{\beta}s$	s	s	$s+m$	s	s
$\boldsymbol{x}$	$R_\phi \boldsymbol{x}$	$\boldsymbol{x}$	$\boldsymbol{x}-\boldsymbol{v}r$	$\boldsymbol{x}-\boldsymbol{w}s$	$\boldsymbol{x}$	$\boldsymbol{x}$	$\boldsymbol{x}+\boldsymbol{a}$

which of course coincides with the action of $\mathscr{U}$ on its homogeneous space $\mathscr{U}/\mathscr{N}$ of $\mathscr{U}$ by the homogeneous subgroup $\mathscr{N}$. A study of the group $\mathscr{U}$ is now under development.

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